Excitations and dynamics of a two-component Bose-Einstein condensate in 1D

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Abstract: We study different solutions and their stability for a two component one-dimensional Bose-Einstein condensate. First of all, we present the spectrum of the ground state in the miscible phase and we identify two families of excitations. Using this spectrum we characterize the transition between the miscible and immiscible phase, and we analyze different solutions which appear in this new phase. With this background, now we study an excited state in the miscible phase, the dark-dark soliton. We present its spectrum and we find two different anomalous modes, one of them, the out-of-phase mode, collides with other modes of the ground state and creates unstable regions. In these regions we find an interesting decay process to two new solutions, dark-antidark solitons. Besides, we study the dynamics of the dark-dark soliton in a harmonic trap, and we find that the oscillation frequency is the same as for a dark soliton in one component one-dimensional Bose-Einstein condensate.

I. INTRODUCTION

Since the realization of the first Bose-Einstein condensate (BEC) a large amount of theoretical and experimental work has been realized (see [1] for a summary). An interesting scenario is the connection between the non-linear waves and atomic systems, that leads to the so-called matter-wave solitons. Specifically, mean field descriptions of BECs (Gross-Pitaevskii equations, GP) incorporate a non-linear term that is proportional to the s-wave scattering length. Depending on the sign and magnitude of the interatomic interaction and the dimensionality of the system, a large number of structures [2] can be found: dark and bright solitons, vortices, vortex rings, etc.

Another interesting aspect of BECs is the study of multicomponent systems, which can be described by a set of coupled GP equations. In recent years a large number of experiments realized mixtures of two condensates using two hyperfine states [3] or two atomic species [4]. These systems are interesting because they present a quantum phase transition from miscible to immiscible. Besides, the study of matter-wave solitons in these systems becomes complex, a new family of structures appear: dark-dark (DD), dark-bright, dark-antidark (DA), etc.

The aim of this work is to study the two component BECs using the Bogoliubov-de Gennes formalism. We characterize the miscible and the immiscible phase and the transition from one to another, we also study a matter-wave soliton in the miscible phase: the DD soliton; proposed by [5] and experimentally observed in [6].

The present work is structured as follows: First of all, in section II we introduce the theoretical model. In section III we present our first results, including the spectrum of the ground state in the miscible phase and the transition between the miscible and immiscible phase. In section IV we study a particular excited state in the miscible phase: the DD soliton. To sum up we present our conclusions in section V.

II. THEORETICAL BACKGROUND

A. Mean Field Description

The dynamics of BECs with two different components at zero temperature can be described, in the mean field regime, by two wave functions $\phi_1(\mathbf{r},t)$ and $\phi_2(\mathbf{r},t)$ that are governed by two coupled GP equations:

$$i\hbar\frac{\partial}{\partial t}\phi_i = \left(-\frac{\hbar^2}{2m_i}\nabla^2 + V_i(\mathbf{r}) + g_{ii}|\phi_i|^2 + g_{ij}|\phi_j|^2\right)\phi_i. \tag{1}$$

In our case the two components are trapped by an axisymmetric harmonic potential: $V_i(\mathbf{r}) = \frac{1}{2}m_i(w_z z^2 + w_\perp(x^2+y^2))$. m_i is the atomic mass of the i-component and $g_{ij} = \frac{2\pi\hbar^2 a_{ij}}{m_{ij}}$ is the interaction (between component i and j) coupling constant with $m_{ij} = \frac{m_i m_j}{m_i+m_j}$. The normalization condition for the wave functions is $\int|\phi(\mathbf{r},t)|^2 d^3\mathbf{r} = N_i$ where N_i is the number of particles of component $i=1,2$.

Sometimes these systems can be done using two different hyperfine states of the same alkaline metal, such as $(F=2, M_F=2)$ and $(F=1, M_F=1)$ of ^{87}Rb [3]; in that case we can set that $m_1 = m_2 = m$. Also, in our work we are going to impose the condition $V_1(\mathbf{r}) = V_2(\mathbf{r}) = V(\mathbf{r})$ meaning that the trap is the same for both components. We will assume that we have a highly anysotropic trap, and so the longitudinal frequency will be much smaller than the transversal frequency ($w_z \ll w_\perp$). A geometry which would reproduce very well this consideration is the elongated cigar-shaped trap [7]. On this basis the transversal degrees of motion are frozen and one can obtain an effective 1D equation for the longitudinal wave function $\phi_i(z,t)$. In this effective equation the external potential is $V(z) = \frac{1}{2}m\Omega^2 z^2$ and the coupling constant is reduced to $g_{ij}^{1D} = \frac{g_{ij}}{2\pi a_\perp^2}$ where $a_\perp = \sqrt{\hbar/mw_\perp}$ is the characteristic length scale of the transversal motion. $\Omega = w_z/w_\perp \ll 1$ is the aspect ratio.

Besides, one can set $\hbar w_\perp$, $a_\perp$ and $w_\perp^{-1}$ as energy, length and time units and thus, recover dimensionless equations replacing the wave function as $\phi_i \to \phi_i\sqrt{N_i/a_\perp}$:

$$i\frac{\partial}{\partial t}\phi_i = \left(-\frac{1}{2}\frac{\partial^2}{\partial z^2} + \frac{1}{2}\Omega^2 z^2 + \bar{g}_{ii}^{(1D)}|\phi_i|^2 + \bar{g}_{ij}^{(1D)}|\phi_j|^2\right)\phi_i. \quad (2)$$

Where $\bar{g}_{ij}^{(1D)} = g_{ij}^{(1D)}N_i/\hbar w_\perp a_\perp = 2N_i a_i/a_\perp$ and now the normalization condition for the wave functions is $\int|\phi_i(z)|^2 dz = 1$. Just for convenience we will set $\bar{g}_{ij}^{(1D)} \equiv g_{ij}$ but remember that this is a reduced and dimensionless coupling constant.

B. Excitation Spectrum

The main objective of our work is to study the stability of some solutions to Eqs. (2). To do that we have obtained the Bogoliubov-de Gennes (BdG) spectrum. These can be obtained considering excitations of a given solution: $e^{-i\mu t}\left(\phi_i + u_i e^{iwt} + v_i^* e^{-iw^* t}\right)$ where w is a general complex eigenfrequency and (u_i, v_i) are the eigenmodes associated to the component i. Introducing this ansatz into Eqs. (2) and collecting linear terms in (u_i, v_i), one obtains the following BdG equations:

$$M\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = w\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} \quad (3)$$

where:

$$M = \begin{pmatrix} h_1 & g_{11}\phi_1^2 & g_{12}\phi_1\phi_2^* & g_{12}\phi_1\phi_2 \\ -g_{11}\phi_1^{*2} & -h_1 & -g_{12}\phi_1^*\phi_2^* & -g_{12}\phi_1^*\phi_2 \\ g_{12}\phi_1^*\phi_2 & g_{12}\phi_1\phi_2 & h_2 & g_{22}\phi_2^2 \\ -g_{12}\phi_1^*\phi_2^* & -g_{12}\phi_1\phi_2^* & -g_{22}\phi_2^{*2} & -h_2 \end{pmatrix}. \quad (4)$$

Here $h_i = -\frac{1}{2}\frac{\partial^2}{\partial z^2} + \frac{1}{2}\Omega^2 z^2 + 2g_{ii}|\phi_i|^2 + g_{12}|\phi_{j\neq i}|^2 - \mu_i$. This is an eigenvalue problem and has a non-trivial solution for the condition $|M - wI| = 0$.

III. MISCIBLE AND IMMISCIBLE PHASE

In this section we characterize the ground state of Eqs. (2) in the miscible phase and compute their BdG excitation spectrum. Besides, we study the transition from the miscible to immiscible phase. The ground state is found using a numerical method (split operator method with relaxation in imaginary time) and later the BdG Eqs. (3) are numerically solved in momentum space. The excited spectrum for the case $g_{12} > 0$ (repulsive intercomponent interaction) is presented in Fig. 1. Here we show the spectrum to the point where the first imaginary eigenfrequency appears (meaning that our state is unstable). It is known that in the homogeneous case the binary system presents a phase transition at the point ($g_{12} = \sqrt{g_{11}g_{22}}$) where the system passes from the miscible to the immiscible phase. Here at $g_{12}^{(c)} \approx 1.018g$ we found the critical point, which is larger. This has a physical interpretation: the trap tends to squeeze the two components and this causes the system to require more repulsive interaction to reach the immiscible phase.

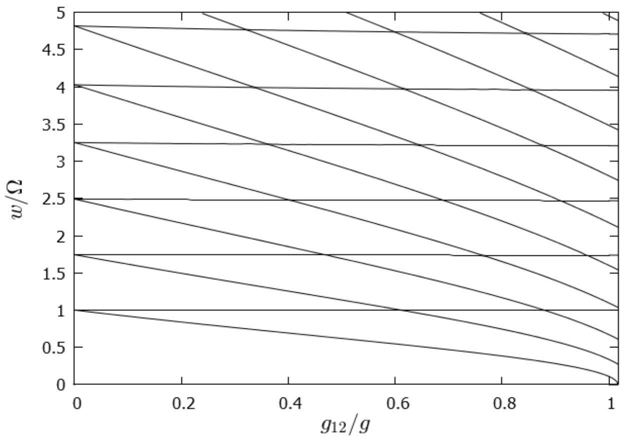

FIG. 1: Numerical results for the BdG spectrum Eqs. (3). Here we present the different modes that appear for the miscible phase. All these modes only have a real part (are stable under excitations). Modes are normalized to the aspect ratio $\Omega = 0.1$ and are represented versus the intercomponent interaction g_{12}/g. When $g_{12} = 0$ the two components have enough intracomponent interaction to be in the Thomas-Fermi limit; this is a condition imposed by us.

Now we are going to interpret the different branches that appear in the spectrum. We can distinguish two different types: the ones that remain constant when the intercomponent interaction increases, and the ones that decrease in a linear way (at low g_{12}) as g_{12} increases. It is important to remember that the spectrum of one component in the Thomas-Fermi limit is analytic for the different modes $w_n = \sqrt{\frac{n(n+1)}{2}}\Omega$ (n=1 dipolar, n=2 quadrupolar, etc) [8]. It is interesting to see that the branches which remain constant obey this relation. Hence, the interpretation is that these are in-phase oscillations under the trap, meaning that the two components move together with a common center of mass.

The other branches are associated to a relative motion between components (out-of-phase oscillations) and they reduce their value as g_{12} increases, because the repulsive interaction between them slows down the movement. In Fig. 2 is represented a time evolution of the two components where initially are separated in different directions so here appears a relative motion. Now we are able to understand why the phase transition is associated with the point where the dipolar mode out-of-phase goes to zero [9].

In the immiscible phase the components are separated in different space regions and there is no relative motion

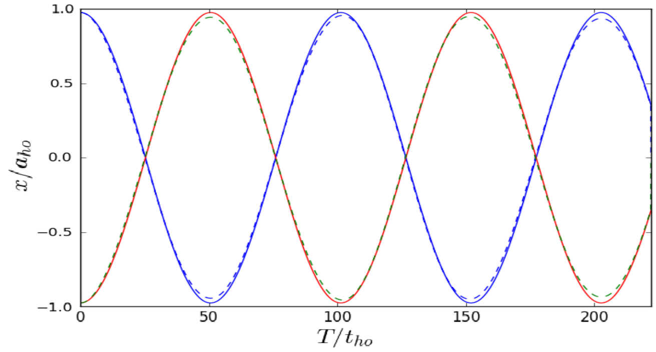

FIG. 2: The dipolar mode out-of-phase is excited separating the two condensates ($\Delta x/a_{ho} \approx 1$) in opposite directions, for $g_{12} = 0.5g$. Here we represent the mean value $< x >$ for each condensate (dashed lines). Also is plotted (continuous lines) a cosinusoidal function with the frequency given by the spectrum in Fig. 1 ($w/\Omega \approx 0.6198$).

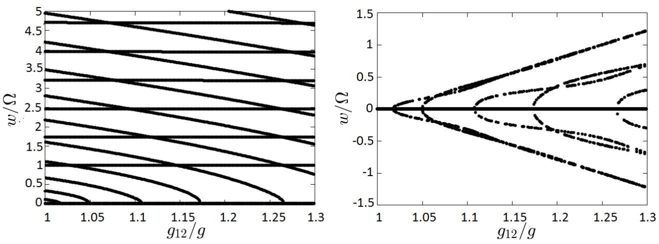

FIG. 3: Extension of the spectrum for the miscible state in the immiscible region. On the left we have represented the real part and on the right the imaginary one. The different modes (dipolar, quadrupolar...) go to zero as g_{12} increases and at the point which are zero a new imaginary branch appears. These different branches are associated to new immiscible states (1-1 hump, 2-1 hump, etc).

between them. So the first stationary state that we can find in the immiscible region is one with the components occupying different regions and with no relative motion (there will be only one interphase).

Our next objective is to study different states that arise in the immiscible phase. In order to do that we calculate a metastable solution where the two components are mixed using the relaxation imaginary time method. With this method we can make the continuation of the spectrum in the miscible phase to the immiscible phase, Fig. 3. For each real mode that goes to zero appears a new non-zero imaginary one, associated to a new possible decay of the miscible state. The different states that appear in the immiscible phase now can be understood as different real modes which are frozen. For instance, when the quadrupole mode is frozen (at $g_{12} \approx 1.05g$) a new state appears with a component with one hump and the other one with two humps (2-1 hump state), as shown in Fig. 4. It is interesting to notice that the mean life time of the miscible phase is related with the inverse of imaginary part maximum. In fact, the only state that is totally stable in the immiscible phase is the 1-1 hump

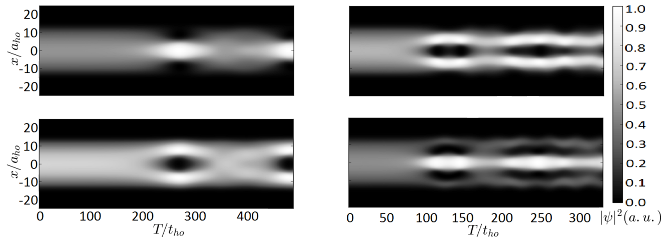

FIG. 4: Real time evolution of the densities (on the top we plot the first component and on the bottom the second one) after applying a very small perturbation to the solution ψ_0, $\psi = \psi_0 + \delta\psi$ where $\delta\psi$ is Gaussian noise with an amplitude of 10^{-2}. On the left we show the decay process for $g_{12} = 1.1g$; and the miscible state decays to the 2-1 hump state. On the right for $g_{12} = 1.2g$ and it decays to the 3-2 hump state. These evolutions go along with the imaginary part in Fig. 3.

state; with a long-time evolution all the immiscible states decay to the 1-1 hump state. The miscible state is the most unstable one and the others with humps will become more stable as g_{12} increases; see [10] for a more extensive discussion.

IV. DARK-DARK SOLITONS

Until now we have studied the ground state in the miscible phase and the different states that appear in the immiscible phase. Now we concentrate on an excited solution in the miscible one, the DD soliton. This solution is characterized by having a dark soliton in each component at the same position. In the one-component system the dark soliton's spectrum is just the same as the ground state solution (without dark soliton) but with an extra mode. This mode is called 'anomalous mode' and is associated with the oscillation frequency of the dark soliton under the trap; in the large chemical potential limit is predicted to be $\Omega/\sqrt{2}$ [2, 11]. Furthermore all the modes are real, so the dark soliton is a stable state.

First of all, we present an example of DD soliton in the Fig 5. Just as the one-component case, the point where the density goes to zero presents a phase-jump of π in the phase of the condensate. Now we are going to solve Eqs.(3) using this solution. In Fig. 7 we present the spectrum for increasing values of g_{12}. It starts from $g_{12} = 0$ where we have the modes associated to one component $w_n = \sqrt{\frac{n(n+1)}{2}}\Omega$ plus the anomalous mode $w = \Omega/\sqrt{2}$. As g_{12} increases, the in-phase and out-of-phase modes appear like in the case without DD soliton Fig. 1. But now the anomalous mode is also divided in two modes: one remains constant with g_{12} just like the in-phase modes of the ground state, that is why we named it the in-phase anomalous mode; the other one increases with g_{12} and we named it the out-of-phase anomalous mode.

The interpretation of the in-phase anomalous mode is a combination of the anomalous mode for the dark soliton in one component (the oscillation frequency under the

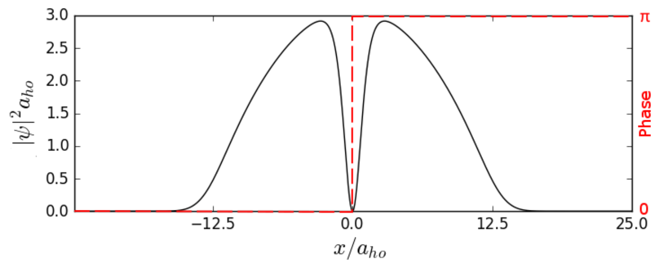

FIG. 5: Example of a DD soliton with parameters $\Omega = 0.1$, $g = 0.2$ and $g_{12} = 0.5g$. The densities and phases of the two components are identical. The solution is found by solving Eqs. (2) with a split operator method and with imaginary time. One has to search excited states starting from a proper ansatz; we used a gaussian profile multiplied by a hyperbolic tangent.

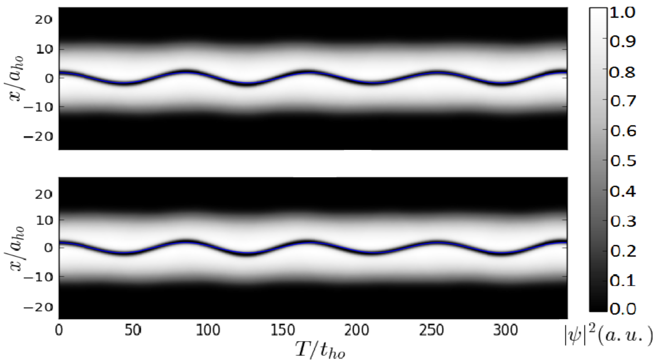

FIG. 6: Real time evolution of the densities (on the top we plot the first component and on the bottom the second one) with parameters $\Omega = 0.1$, $g = 0.2$ and $g_{12} = 0.5g$. The DD soliton is printed outside the center of the trap and the solution is found using the imaginary time method. Also is plotted a cosinusoidal function (blue line) with frequency $\Omega/\sqrt{2}$.

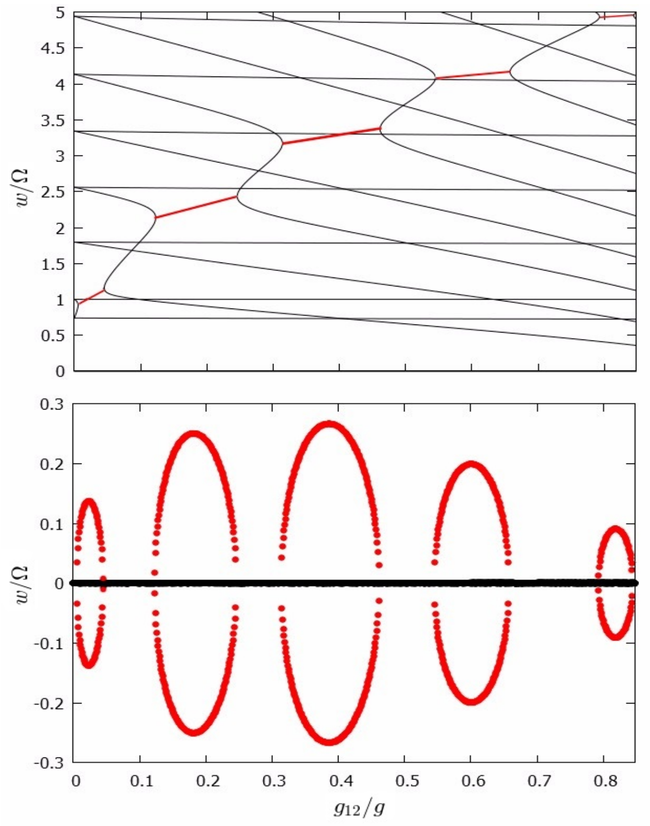

FIG. 7: Numerical results for the BdG spectrum Eqs. (3) using as a solution the DD soliton. On the top is represented the real part and on the bottom the imaginary one. Modes are normalized to the aspect ratio $\Omega = 0.1$ and are represented versus the intercomponent interaction (g_{12}/g where $g = 0.2$). When $g_{12} = 0$ the two components have enough intracomponent interaction to be in the Thomas-Fermi limit. We remark (red lines) the unstable regions.

trap) and the in-phase oscillations (the movement of the two components with a common center of mass). Hence the in-phase anomalous mode will give us the oscillation frequency of the DD soliton under the trap, and just like the in-phase oscillations of the ground state, it does not depend on g_{12}. The oscillatory movement of the DD soliton is presented in Fig. 6. The out-of-phase anomalous mode also starts from the value $\Omega/\sqrt{2}$ but rapidly increases as g_{12} increases, as opposed to the out-of-phase modes of the ground state. It is important to remember that the decrease of the out-of-phase modes was associated to the repulsive nature of the intercomponent interaction. Now the dip of the dark soliton in one component creates an effective attractive potential at the same point in the other component, hence the interaction between the two dark solitons at different components will be attractive (because of the repulsive intercomponent interaction). Due to this effect the DD soliton solution exists for $g_{12} > 0$ but for negative values the effective interaction would be repulsive (if $g > 0$) and the solution does not exist.

Until now we have not discussed the stability of the solution in this region; this can be explored looking for modes with imaginary part. In Fig. 7 we present the imaginary part of the spectrum and we observe different unstable regions; where the DD soliton is unstable, as seen in Fig. 8. In the decay process the DD soliton starts emitting some radiation that decouples it and this creates a relative motion of two dark solitons in each component. As we commented before, the dip of the dark soliton creates an effective attractive potential in the other component. Now this interaction attracts some atoms of the other component in the position where the dark soliton is, and this process creates a new kind of structure: the dark-antidark (DA) soliton, recently studied in [12].

Now we concentrate on the origin of these instabilities. If one checks the real part of the spectrum in Fig. 7 one can observe that the out-of-phase anomalous mode collides with the out-of-phase modes of the ground state and this leads to a resonant frequency; this is called a Hopf bifurcation and was studied in the context of one component systems in [2]. The Hopf bifurcation is char-

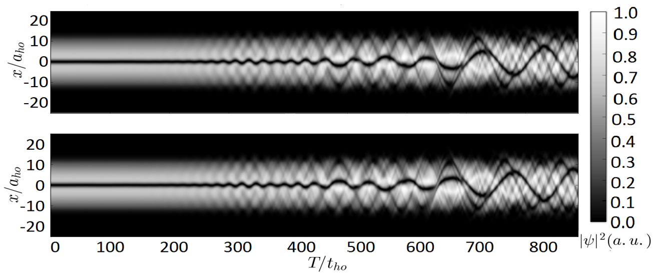

FIG. 8: Real time evolution of the densities (on the top we plot the first component and on the bottom the second one) with parameters $\Omega = 0.1$, $g = 0.2$ and $g_{12} = 0.4g$. The DD soliton is printed in the center of the trap and the solution ψ_0 is found using the imaginary time method. It is applied a very small perturbation $\delta\psi$ to the solution, where $\delta\psi$ is Gaussian noise with an amplitude of 10^{-2}. Due to the instabilities the DD soliton decays into two DA solitons emitting radiation.

acterized by the collision of two real eigenvalues which creates a complex eigenvalue and its complex conjugate, leading to a region of instability. Increasing g_{12} the complex eigenvalues become to two pure real eigenvalues and the instability disappears. In our case of two components we can observe that the out-of-phase anomalous mode only collides with the out-of-phase modes of the ground state, not with the in-phase modes. Looking closely to the spectrum one can check that there is no collision with all the out-of-phase modes, only with the odd ones. This nature of the Hopf bifurcations in the context of two components BEC is not very well understood but this leads to the characteristic instability regions of Fig. 7.

V. CONCLUSIONS

We have studied the stability and dynamics of BECs with two different components under an asymmetric harmonic trap. We have used the BdG formalism to study the stability of different solutions. First of all we have presented the ground state in the miscible phase and its BdG spectrum, and we have characterized its different modes associated with two movements: in-phase and out-of-phase oscillations. We have observed that when the first out-of-phase mode approaches zero an instability appears and the miscible state becomes unstable; this is associated with the miscible-immiscible transition. Besides, we have constructed a continuation of the spectrum in the immiscible region and, with this, we have characterized different states with humps studying the decay of the miscible state. We have have related the tendency of the out-of-phase modes to go to zero (this creates new imaginary modes) and the emergence of new states with immiscible properties (1-1 hump state, 2-1 hump state...).

On the other hand, we have presented an excited state in the miscible phase, the DD Soliton. We have calculated its BdG spectrum and have identified two different anomalous modes: in-phase and out-of-phase. The in-phase mode remains constant with g_{12}, and is associated with the oscillation of the solitons at different components with a common center of mass; its value is the same as for a dark soliton in one component. The out-of-phase mode increases as g_{12} increases and is associated with the relative motion of the dark solitons. The collision between the out-of-phase anomalous mode and the odd out-of-phase modes of the ground state, creates complex frequencies that lead to unstable regions. Hence, we have presented different regions of stability where the DD solution could be observed and will be stable; it would also be possible to observe its dynamics under the trap. Finally, the decay process of the DD soliton is presented, and we have observed that it decays to two DA solitons, in each component, and they effectuate a relative motion between them. This could be a technique to explore the dynamics of two DA solitons. Besides, it would be interesting to characterize the spectrum of two stationary DA solitons and to compare the frequency associated to the relative motion with the decay presented in this work.

Acknowledgments

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